ON THE UNION STABILIZATION OF
TWO HEEGAARD SPLITTINGS

JUNG HOON LEE

ABSTRACT. Let two Heegaard splittings $V_1 \cup W_1$ and $V_2 \cup W_2$ of a 3-manifold M be given. We consider the *union stabilization* $M = V \cup W$ which is a common stabilization of $V_1 \cup W_1$ and $V_2 \cup W_2$ having the property that $V = V_1 \cup V_2$. We show that any two Heegaard splittings of a 3-manifold have a union stabilization. We also give some examples with numerical bounds on the minimal genus of union stabilization. On the other hand, we give an example of a candidate for which the minimal genus of union stabilization is strictly larger than the usual stable genus — the minimal genus of common stabilization.

1. INTRODUCTION

A *Heegaard splitting* $M = H_1 \cup_S H_2$ of a 3-manifold M is a decomposition of M into two handlebodies H_1 and H_2 and it is well known that every compact 3-manifold admits Heegaard splittings.

A *stabilization* of a Heegaard splitting $H_1 \cup_S H_2$ is an operation that results in a new Heegaard splitting $H_1' \cup_{S'} H_2'$, with the genus increased by one. Here H_1' is obtained by adding trivial 1-handle to H_1 whose core is ∂-parallel in H_2 and H_2' is obtained by removing the 1-handle from H_2.

When two Heegaard splittings $V_1 \cup W_1$ and $V_2 \cup W_2$ of a 3-manifold are given, they become isotopic after a sequence of stabilizations [8]. The minimal genus among the common stabilizations is called the *stable genus*. Now we introduce a new concept slightly stronger than the common stabilization.

A Heegaard splitting $V \cup W$ is a *union stabilization* of $V_1 \cup W_1$ and $V_2 \cup W_2$ if

- $V \cup W$ is a common stabilization of $V_1 \cup W_1$ and $V_2 \cup W_2$.
- $V = V_1 \cup V_2$

In section 2, we show that any two Heegaard splittings have a union stabilization, similar to the results of [8]. The minimal genus among the union stabilizations is called the *union genus*.

Remark 1.1. (1) *From the definition of the union stabilization, it follows that* $W = W_1 \cap W_2$.

---

(2) *The union stabilization and the union genus can be defined similarly for Heegaard splittings of 3-manifolds with boundary.*

When two Heegaard splittings of a 3-manifold are given, it has been conjectured that they become isotopic after a single stabilization of the larger genus one [5]. However, recently there are results that this is not true and there are examples that require as many stabilizations as the genus of the Heegaard splittings [2], [3], [4], [1]. This implies that the union genus also can be large enough. In section 3, we give some examples with bounds on the union genus.

In section 4, we give an example of a candidate for which the union genus is strictly larger than the stable genus implying that a common stabilization of two Heegaard splittings cannot be obtained by the union of both handlebodies.

2. EXISTENCE OF THE UNION STABILIZATION

We begin with a simple case where two Heegaard splittings are isotopic.

Proposition 2.1. *Suppose $V_1 \cup W_1$ and $V_2 \cup W_2$ are isotopic Heegaard splittings of a 3-manifold. Then there is a union stabilization with genus increased by one.*

Proof. The two handlebodies of Heegaard splittings are yellow and blue handlebodies respectively (Fig. 1). Let some parts of each handlebodies coincide and it is colored green. We can see that the disk D serves as a stabilizing disk for both.   □
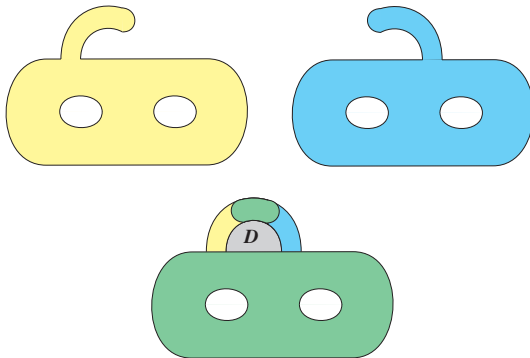

FIGURE 1. Yellow + Blue = Green

A *spine* Σ_H of a handlebody H is a finite graph in H where H defomation retracts to Σ_H.

The following is a result similar to [8].

Theorem 2.2. *Let $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ be two Heegaard splittings of a 3-manifold M. Then there exists a union stabilization of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$.*

Proof. Take V_1 and V_2 as a very thin neighborhood of Σ_{V_1} and Σ_{V_2} respectively. So we may assume that $V_1 \cap V_2 = \emptyset$. Hence V_2 is in W_1. Furthermore, we assume that V_2 is an embedded graph in W_1. Note that genus g handlebody is homeomorphic to g-punctured disk $\times I$. We consider the projection of V_2 to the g-punctured disk and the diagram of V_2 with over and under information.

If we add sufficiently many tunnels to V_2 (at least, as many tunnels as the number of crossings of the diagram), the exterior in W_1 would be a handlebody. Connect each tunnel to ∂W_1 with an arc so that the exterior of the union of V_2 and tunnels and arcs is still a handlebody. However, adding tunnels and arcs is equivalent to the isotopy of some parts of V_2 along the tunnels and arcs and let the parts coincide with some parts of V_1. Fig. 2 shows the isotopy. Then we can see that $(V_1 \cup V_2) \cup cl((V_1 \cup V_2)^c)$ is a union stabilization since $V_1 \cap V_2$ is a collection of 3-balls and by the uniqueness of Heegaard splittings of a handlebody [7]. □
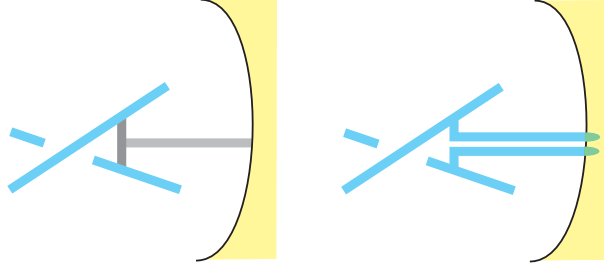

FIGURE 2. Isotopy of V_2 and $V_1 \cap V_2 =$ green 3-balls

3. EXAMPLES

In this section we show some examples with numerical bounds on the union genus. First example is that the spine of the handlebody of one Heegaard splitting is on the other Heegaard surface. For a Heegaard splitting $V \cup_S W$, let $g(S)$ denote the genus of S.

Proposition 3.1. *Let $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ be two Heegaard splittings of a 3-manifold M such that Σ_{V_2} is in S_1. Then the union genus of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is less than or equal to $g(S_1) + g(S_2)$.*

Proof. Take V_2 as s a very thin neighborhood of Σ_{V_2}. Push V_2 slightly into W_1 so that there is an annulus of parallelism between ∂V_2 and ∂V_1. Connect V_1 and V_2 by an essential arc of the annulus. Then the result is a $g(S_2)$-times stabilization of $V_1 \cup_{S_1} W_1$. Connecting V_1 and V_2 by an arc is equivalent to making some parts of V_1 and V_2 coincide as in the proof of Theorem 2.2. Then since $V_1 \cap V_2$ is a 3-ball and by the uniqueness of Heegaard splittings of a handlebody [7], it is also a $g(S_1)$-times stabilization

of $V_2 \cup_{S_2} W_2$. Hence the union genus of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is less than or equal to $g(S_1) + g(S_2)$. $\square$

The next example is the (g,n) presentation of a knot K in S^3. Let $V_1 \cup V_2$ be a decomposition of S^3 into two standardly embedded genus g handlebodies V_1 and V_2. Suppose that $V_i \cap K$ $(i = 1, 2)$ are trivial n-string tangles. Then we call $(V_1 \cap K, V_2 \cap K)$ the genus g, n-bridge presentation of K, or (g, n) presentation for short. Clearly such a knot K has tunnel number less than or equal to $g + n - 1$ as Fig.3 illustrates.
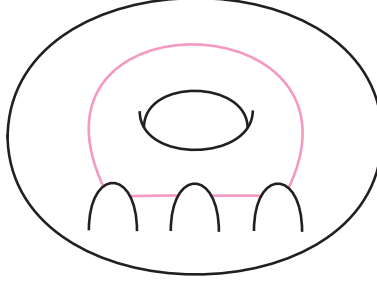

FIGURE 3. (g, n) presentation of a knot with a tunnel system $(g = 1,\ n = 3)$

Proposition 3.2. *Suppose a knot K with a (g, n) presentation has tunnel number $g + n - 1$. Let $\mathcal{T}_i$ be a tunnel system in V_i as in Fig. 3 $(i = 1, 2)$. Then the two Heegaard splittings induced by $\mathcal{T}_1$ and $\mathcal{T}_2$ has union genus less than or equal to $2g + 2n - 1$.*

Proof. It suffices to show that the exterior in S^3 of the union of K and the tunnels in $\mathcal{T}_1$ and $\mathcal{T}_2$ is a genus $2g + 2n - 1$ handlebody. The exterior $X_1 \subset V_1$ of the union of the trivial n-string tangle and $\mathcal{T}_1$ is a genus $2g + 2n - 1$ handlebody. The exterior $X_2 \subset V_2$ of the union of the trivial n-string tangle and $\mathcal{T}_2$ is homeomorphic to a $(2n$-punctured genus g surface$) \times I$. Since $X_1 \cap X_2$ is a $2n$-punctured genus g surface, the total exterior $X_1 \cup X_2$ is homeomorphic to a genus $2g + 2n - 1$ handlebody. $\square$

4. TOWARDS AN EXAMPLE THAT UNION GENUS > STABLE GENUS

In this section, we construct an example of a candidate for which the union genus is strictly larger than the stable genus.

Consider a 2-bridge knot and two unknotting tunnels as in the Fig. 4, which are put on the bridge sphere [6].

The two unknotting tunnels intersect in two points. By some perturbation, we can make them disjoint. Then there exist some knot, for example a 2-bridge knot 6_3 in knot table, such that the four cases of perturbations as in Fig. 5 do not give genus three handlebodies. (For each case, we can check the fundamental group of the exterior using the Wirtinger presentation.)
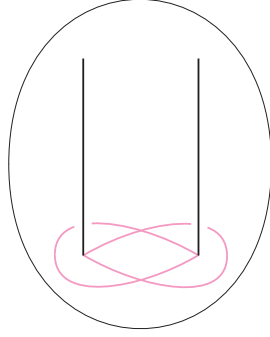
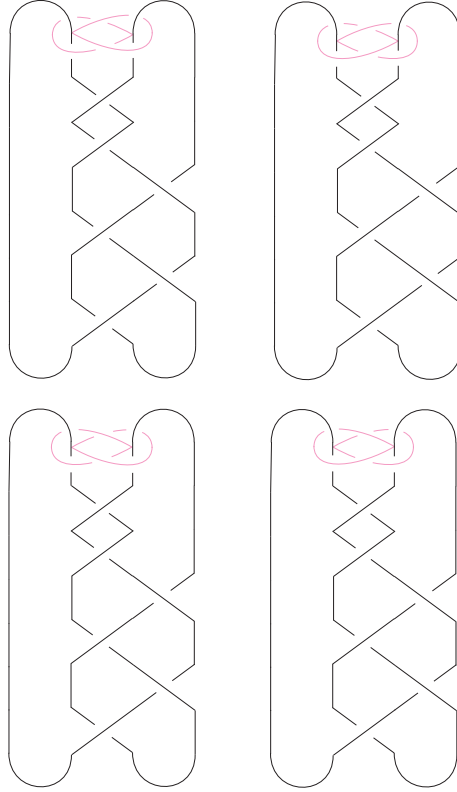FIGURE 4. Two unknotting tunnels of a 2-bridge knot

FIGURE 5. These do not give genus three Heegaard splittings.

We guess that the two Heegaard splittings of the 2-bridge knot 6_3 induced by the unknotting tunnels in Fig. 4 and Fig. 5 have union genus four.

Remark 4.1. *We can see that the stable genus of above example is three. The common stabilization is isotopic to the stabilization of the genus two Heegaard splitting induced by the tunnel connecting the two strands of the knot (Fig. 6).*
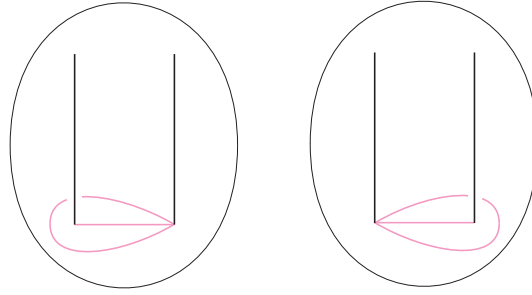
FIGURE 6. The stable genus is three.

So finally we have the following question.

Question 4.2. *There exists two Heegaard splittings of a 3-manifold that the union genus is strictly larger than the stable genus.*

REFERENCES

[1] D. Bachman, *Stabilizations of Heegaard splittings of sufficiently complicated 3-manifolds (Preliminary Report)*, arXiv:math.GT/0806.4689.

[2] J. Hass, A. Thompson and W. Thurston, *Stabilization of Heegaard splittings*, arXiv:math.GT/0802.2145.

[3] J. Johnson, *Flipping and stabilizing Heegaard splittings*, arXiv:math.GT/0805.4422.

[4] J. Johnson, *Bounding the stable genera of Heegaard splittings from below*, arXiv:math.GT/0807.2866.

[5] R. Kirby, *Problems in low-dimensional topology*, preprint, http://math.berkeley.edu/∼kirby.

[6] T. Kobayashi, *Classification of unknotting tunnels for two bridge knots*, Proceedings of the 1998 Kirbyfest, Geometry and Topology Monographs 2 (1999), 259–290.
5 to appear in Journal of Knot Theory and its Ramifications.

[7] M. Scharlemann and A. Thompson, *Heegaard splittings of $(surface) \times I$ are standard*, Math. Ann. **295** (1993) 549–564.

[8] J. Singer, *Three-dimensional manifolds and their Heegaard diagrams*, Transactions of the American Mathematical Society, Vol. **35**, No. 1. (Jan., 1933), 88–111.

SCHOOL OF MATHEMATICS, KIAS, 207-43, CHEONGNYANGNI 2-DONG, DONGDAEMUN-GU, SEOUL, KOREA,
 E-mail address: jhlee@kias.re.kr